Math 347 Worksheet
Lecture 10: Properties of functions
September 21, 2018

1) Recall from Homework 1 that a subset $S \subseteq A \times B$ defines a function $f : A \to B$, only if for all $a \in A$ the set $S \cap p_1^{-1}(a)$ has a single element, where $p_1 : A \times B \to A$ is $p_1((a, b)) = a$. Find a (further) condition on S that guarantees that:
   - (i) f is injective;
   - (ii) f is surjective;
   - (iii) f is bijective.

2) Consider a function f from $\mathbb{R}$ to itself. Prove that if f is strictly monotone[1] then f is injective. Is f surjective as well?

3) Let $S = \{1, \ldots, n\}$, consider A the set of subsets of S with an even number of elements and B the set of subsets of S with an odd number of elements. Prove that $|A| = |B|$, i.e. give a bijection between A and B.

4) Given real numbers a, b, c and d consider the following function
$$f(x, y) = (ax + b, cx + d)$$
from $\mathbb{R}^2$ to $\mathbb{R}^2$. Prove that f is injective if and only if f is surjective.

5) Given $f : A \to B$, $g : B \to C$ and $h = g \circ f$, prove or find a counter-example to the following statements:
   - (i) if h is injective, then f is injective;
   - (ii) if h is injective, then g is injective;
   - (iii) if h is surjective, then f is surjective;
   - (iv) if h is surjective, then g is surjective.

6) Let f be a function from a set A to itself. Suppose that A is a finite set. Prove that f is injective if and only if f is surjective. Show that this fails if A is infinite.

7) Show that $f : \mathbb{Z} \to \mathbb{N}$ as defined below is bijective:
$$f(n) = \begin{cases} 2n & \text{if } n \geq 0, \\ -2n - 1 & \text{if } n < 0. \end{cases}$$

8) Consider f, g, h functions from a set A to itself. We say that a function f has the *left cancelation property* if
$$f \circ g = f \circ h \quad \Rightarrow \quad g = h.$$
Analogously, f has the *right cancelation property* if
$$g \circ f = h \circ f \quad \Rightarrow \quad g = h.$$
   - (a) Prove that f has the left cancelation property if f is injective.
   - (b) Is it the case that f has the left cancelation property only if f is injective?
   - (c) Prove that f has the right cancelation property if f is surjective.
   - (d) Is it the case that f has the right cancelation property only if f is surjective?

[1] Look up the definition of that if you need to.